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On the properties of some special functions related to Bessel's functions and their application in heat exchanger theory

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Abstract—Basic properties of $Bs_n(x, y)$ and $Bes_n(x, y)$ functions related to Bessel functions, are presented. The functions are defined by double power series:

$$Bs_n(x,y) \overset{\text{df}}{=} \sum_{m=\max(0,n)}^{\infty} \frac{x^m}{m!} \sum_{k=0}^{m-n} \frac{y^k}{k!}$$

$$Bes_n(x,y) \overset{\text{df}}{=} \sum_{m=\max(-n,0)}^{\infty} \frac{x^{m+n}y^m}{(m+n)!m!}.$$

Numerous formulae are given on the origin of identities, differential formulae as well as the ones for calculation of integrals, whose sub-integral functions comprise the foregoing special functions. Some applications of $Bs_n(x, y)$ and $Bes_n(x, y)$ functions are also given with regard to the theory of cross-flow recuperator and to heat transfer analysis of gas-cooled clinker beds. Some cases of boundary conditions and initial boundary conditions are considered. The solutions are of the so-called closed form which is by far better than those achieved by approximate methods. $Bs_n(x, y)$ and $Bes_n(x, y)$ functions may be applied to theoretical analyses of heat and mass exchangers, regenerators, ion exchangers and for different kinds of heavy equipment used in the chemical industry. These functions can also be applied to control and protect various physical processes.

1. INTRODUCTION

THE SYSTEM of linear partial differential equations in the following general form

$$\sum_{i=1}^{N} c_{ki} \cdot \frac{\partial T_k(x_1, x_2, \ldots, x_N)}{\partial x_i} = \sum_{i=1}^{M} a_{ki} \cdot T_k(x_1, x_2, \ldots, x_N) + b_k(x_1, x_2, \ldots, x_N), \quad k = 1, 2, \ldots, M \tag{1}$$

is of essential importance in the theory of heat and mass exchangers, recuperators, regenerators and of chemical devices. In practice, engineers usually have to deal with devices where heat- or mass-transfer phenomena take place between two media only. In such cases, the above differential equations system reduces to the form

$$\begin{aligned} \frac{\partial T_1^*}{\partial x_1^*} &= a_{11}^* T_1^*(x_1^*, x_2^*) + a_{12}^* T_2^*(x_1^*, x_2^*) + b_1^*(x_1^*, x_2^*) \\ \frac{\partial T_2^*}{\partial x_2^*} &= a_{21}^* T_1^*(x_1^*, x_2^*) + a_{22}^* T_2^*(x_1^*, x_2^*) + b_2^*(x_1^*, x_2^*). \end{aligned} \tag{2}$$

One should observe that, the analyses of heat exchangers in unsteady conditions and of regenerators boil down to the solution of the following equations system

$$\begin{aligned} c_{11} \frac{\partial T_1}{\partial x_1} + c_{12} \frac{\partial T_1}{\partial x_2} &= a_{11} T_1(x_1, x_2) + a_{12} T_2(x_1, x_2) + b_1(x_1, x_2) \\ c_{21} \frac{\partial T_2}{\partial x_1} + c_{22} \frac{\partial T_2}{\partial x_2} &= a_{21} T_1(x_1, x_2) + a_{22} T_2(x_1, x_2) + b_2(x_1, x_2). \end{aligned} \tag{3}$$

However, if the matrix $[c_{ki}]$ is not singular then the above system may be reduced [1, 2] to system (2). The realization of the appropriate transformation of independent variables requires also the correct transformation of initial boundary conditions associated with system (3).

The RHSs of equations (2) do not contain the derivatives of the sought functions and the LHSs contain only the partial derivatives of the first order. In certain particular cases (parallel and counter-current heat exchangers in steady-state conditions), these derivatives become the ordinary derivatives because $x_1^* = x_2^* = x^*$.

The equation system (2) is usually solved by means of Picard's successive approximation method. This method

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NOMENCLATURE

- a parameter in balance equations system of cross-flow recuperator [6]
- a_{ki} constant coefficients
- A_i constant coefficients
- b parameter in balance equations system of cross-flow recuperator [6]
- b_k given functions of appropriate coordinates
- Bes_n family of functions determined by power series (5)
- Bs_n family of functions determined by power series (4)
- B_i constant coefficients
- c_{ki} constant coefficients
- I_n nth order modified Bessel function of the first kind
- J_n nth order ordinary Bessel function of the first kind
- T_k seeking functions (temperatures), in case of cross-flow recuperator
- T_1 temperature value of heating medium
- T_2 temperature value of cooling medium
- x_i coordinates.

Greek symbols

- θ_1 temperature of clinker beds
- θ_2 temperature of gas medium
- κ constant coefficient
- $\bigwedge_i$ quantificator: for each value of i
- μ integration variable, constant coefficient in balance equations system of gas-cooled clinker beds [1]
- ξ, η coordinates.

Superscript

- * notation of variable or function after appropriate linear transformation.

may be applied when the RHSs satisfy Lipschitz's condition with respect to the sought functions. This condition is usually satisfied in the theory of the above-mentioned devices.

Before the application of the successive approximation method, equations (2) have to be put into the form of Volterra integral equations of the second kind. If the RHSs of the different equations are linear with respect to the sought functions and the coefficients are bounded then there exists the matrix function resembling the Green or Riemann functions or resolvent. The convolution of this function with the free term of the Volterra equation constitutes the solution of the equations system considered above. However, in the case when the coefficients are constant and the equations are linear, the function may be calculated effectively and constitutes the matrix power series which is convergent for all values of real or complex variables.

The method of successive approximations is of general meaning and is based on the assumption that the RHSs of differential equations satisfy the Lipschitz condition with respect to sought functions. This method is rather work-absorptive and provides serious difficulties when one tries to write down the obtained matrix series in a general form.

In simpler cases, e.g. when $b_i^*(x_1^*, x_2^*) = 0$, it happens to apply the Laplace transformation [3, 4]. However, even by relatively simple forms of initial boundary conditions one obtains the solution in the form of integrals, which up to now has to be calculated numerically.

In refs. [1, 2, 5, 6] are presented the solutions of differential equations considered here for different forms of initial boundary conditions. These solutions are based on the calculus of Mikusiński operators [7]. The creator of this calculus substantiated it on the basis of abstract algebra outgoing from the properties of convolution and Titschmarsh's theorem. This substantiation is simpler than the theory of integral transformations and assures uniqueness of operations on the functions being integrable in the Lebesgue sense.

The solutions mentioned above contain two families of some special functions $Bs_n(x, y)$ and $Bes_n(x, y)$ defined by means of a double power series of the form

$$Bs_n(x,y) \overset{\text{df}}{=} \sum_{m=\max(0,n)}^{\infty} \frac{x^m}{m!} \sum_{k=0}^{m-n} \frac{y^k}{k!} = \sum_{k=n}^{\infty} Bes_k(x,y) \tag{4}$$

$$Bes_n(x,y) \overset{\text{df}}{=} \sum_{m=\max(-n,0)}^{\infty} \frac{x^{m+n} y^m}{(m+n)!\,m!} = Bs_n(x,y) - Bs_{n+1}(x,y). \tag{5}$$

Both these families of functions are defined for each integer n. They are related to Bessel functions. In regard of this fact, we have introduced the foregoing nomenclature for them. The demonstrated power series are absolutely and nearly monotonously convergent in the Cartesian product $Z_1 \times Z_2$ of the two open complex planes of variables z_1 and z_2. The majority function of Bs_n and Bes_n functions is the series of the exponential function $\exp(|z_1| + |z_2|)$. The above properties are automatically the same for the function of real variables.

The family of functions (4) is associated as well with the functions $\gamma_n(x, y)$ [8]. The appropriate relations are

$$\gamma_{2n}(x,y) = \frac{1}{2}\left[Bs_{2n}\left(\frac{x}{2}, \frac{y^2}{2x}\right) + Bs_{2n}\left(-\frac{x}{2}, -\frac{y^2}{2x}\right)\right]$$
$$\gamma_{2n+1}(x,y) = \frac{1}{2}\left[Bs_{2n}\left(\frac{x}{2}, \frac{y^2}{2x}\right) - Bs_{2n}\left(-\frac{x}{2}, -\frac{y^2}{2x}\right)\right]. \quad (6)$$

The introduction of $Bs_n(x, y)$ and $Bes_n(x, y)$ functions permits us to obtain certain solutions in the so-called closed forms [1, 2, 6] of the differential equations system considered here. It is also possible to obtain the same solutions when the initial boundary conditions are different in form than those presented in this paper.

In order to apply the Bes_n and Bs_n functions to theoretical analyses of heat and mass exchangers, recuperators, regenerators, ion exchangers and for different kinds of heavy equipment used in the chemical industry and in the control and protection of various physical processes, it is necessary to know the basic properties of those which are presented in successive sections.

2. PROPERTIES OF FAMILIES OF $Bes_n(x, y)$ AND $Bs_n(x, y)$ FUNCTIONS

In this section we adduce basic properties of the families of $Bes_n(x, y)$ and $Bs_n(x, y)$ functions. Numerous identities, differential formulae and formulae for finding the primary functions of some class of functions including $Bes_n(x, y)$ and $Bs_n(x, y)$ functions are given.

2.1. *Identities*

Since series (4) and (5) are absolutely convergent, the order of summation may be changed. This property yields the following identities

$$Bes_n(x,y) \equiv \begin{cases} \left(\frac{x}{y}\right)^{n/2} \cdot I_n(2\sqrt{(xy)}), & \text{for} \quad xy > 0 \\ \left(-\frac{x}{y}\right)^{n/2} \cdot J_n(2\sqrt{(-xy)}), & \text{for} \quad xy < 0 \end{cases}$$
$$Bes_n(x,y) \equiv Bes_{-n}(y,x) \quad (7)$$
$$x^n \cdot Bes_{-n}(x,y) \equiv y^n \cdot Bes_n(x,y), \quad \text{for} \quad n \geqslant 0 \quad \text{or} \quad xy \neq 0$$
$$\alpha^n \cdot Bes_n(x,y) \equiv Bes_n\left(\alpha x, \frac{y}{\alpha}\right), \quad \text{for} \quad \alpha \neq 0$$
$$\alpha^n \cdot Bes_{-n}(\alpha x, y) \equiv Bes_n(\alpha y, x), \quad \text{for} \quad \alpha \neq 0 \quad \text{or} \quad n \geqslant 0$$

and

$$Bs_n(x,y) + Bs_{-(n-1)}(y,x) \equiv e^{x+y}$$
$$Bs_n(x,y) + Bs_n(y,x) \equiv \begin{cases} e^{x+y} - \sum_{k=-(n-1)}^{n-1} Bes_k(x,y), & \text{for} \quad n \geqslant 1 \\ e^{x+y} + \sum_{k=n}^{k=-n} Bes_k(x,y), & \text{for} \quad n \leqslant 0 \end{cases} \quad (8)$$
$$Bs_{n-l}(x,y) - Bs_{n+m}(x,y) \equiv \sum_{k=-l}^{m-1} Bes_{n+k}(x,y), \quad \text{for} \quad l, m \geqslant 0$$
$$Bs_{2n-1}(x,y) + Bs_{2n-1}(-x,-y) \equiv Bs_{2n}(x,y) + Bs_{2n}(-x,-y)$$
$$Bs_{2n+1}(x,y) - B_{2n+1}(-x,-y) \equiv Bs_{2n}(x,y) - Bs_{2n}(-x,-y).$$

The subsequent identities may be obtained in a different way. The system of differential equations (2) is symmetrical with respect to the change of both the first index of a^*_{ki} and b^*_k and vice versa. Simultaneously there exists the same symmetry with respect to the change of independent variables x^*_k. Hence we have an analogous symmetry of the solutions.

The following identities result from the symmetry and the invariance of system (2) with respect to translations of the coordinate system

$$
\begin{aligned}
Bes_0(x_1+x_2,y) &\equiv Bes_0(x_1,y)+\int_0^y Bes_0(x_1,\eta)\cdot Bes_1(x_2,y-\eta)\,\mathrm{d}\eta \\
Bes_1(x_1+x_2,y) &\equiv Bes_1(x_1,y)+Bes_1(x_2,y)+\int_0^y Bes_1(x_1,\eta)\cdot Bes_1(x_2,y-\eta)\,\mathrm{d}\eta \\
Bes_0(x_1+x_2,y_1+y_2) &\equiv Bes_0(x_1+x_2,y_1)+\int_0^{x_2} Bes_1(y_2,\xi)\cdot Bes_0(y_1,x_1+x_2-\xi)\,\mathrm{d}\xi \\
&\quad +\int_0^{y_2} Bes_0(x_2\,\eta)\cdot Bes_1(x_1,y_1+y_2-\eta)\,\mathrm{d}\eta \\
Bes_1(x_1+x_2,y_1+y_2) &\equiv Bes_1(x_1,y_1+y_2)+\int_0^{y_2} Bes_1(x_2,\eta)\cdot Bes_1(x_1,y_1+y_2-\eta)\,\mathrm{d}\eta \\
&\quad +\int_0^{x_2} Bes_0(y_2,\xi)\cdot Bes_0(y_1,x_1+x_2-\xi)\,\mathrm{d}\xi \\
Bes_1(y,x_1+x_2) &\equiv \int_0^y Bes_0(x_1,\eta)\cdot Bes_0(x_2,y-\eta)\,\mathrm{d}\eta \\
Bes_2(x,y_1+y_2) &\equiv \int_0^x Bes_1(\xi,y_1)\cdot Bes_0(x-\xi,y_2)\,\mathrm{d}\xi \\
Bes_0(x_2,y_2) &\equiv Bes_0(x_1,y_1)+\int_0^{y_2} Bes_0(x_1,y_1+\eta)\cdot Bes_1(x_2-x_1,y_2-\eta)\,\mathrm{d}\eta \\
&\quad -\int_0^{x_1} Bes_{-1}(\xi,y_1)\cdot Bes_0(x_2-\xi,y_2)\,\mathrm{d}\xi \\
Bes_1(x_2,y_2) &\equiv \int_0^{x_2} Bes_0(x_1+\xi,y_1)\cdot Bes_0(x_2-\xi,y_2-y_1)\,\mathrm{d}\xi \\
&\quad -\int_0^{y_1} Bes_1(x_1,\eta)\cdot Bes_1(x_2,y_2-\eta)\,\mathrm{d}\eta.
\end{aligned}
\tag{9}
$$

The identities above are used in various transformations when constructing the final forms of the solutions of system (2). They are also used when finding the primary functions of some class of functions including the $Bes_n(x, y)$ and $Bs_n(x, y)$ functions.

2.2. *Differential formulae*

The formulae for the differentiation of the considered families of functions are listed below

$$
\begin{aligned}
\frac{\partial Bs_n(\alpha x,y)}{\partial x} &= \alpha\cdot Bs_{n-1}(\alpha x,y) \\
\frac{\partial Bs_n(x,\beta y)}{\partial y} &= \beta\cdot Bs_{n+1}(x,\beta y)
\end{aligned}
\tag{10}
$$

and

$$
\begin{aligned}
\frac{\partial Bes_n(\alpha x,y)}{\partial x} &= \alpha\cdot Bes_{n-1}(\alpha x,y) \\
\frac{\partial Bes_n(x,\beta y)}{\partial y} &= \beta\cdot Bes_{n+1}(x,\beta y).
\end{aligned}
\tag{11}
$$

2.3. *Formulae for finding primary functions*

In this section a number of formulae for finding the primary functions of a class of functions including the families $Bes_n(x, y)$ and $Bs_n(x, y)$ are given. The formulae below can be implied from identities (7) and (8) and differential formulae (10) and (11). The integration constants are omitted here.

When subintegral functions are expressed by means of the $Bes_n(x, y)$ functions, we obtain the following formulae

$$\begin{aligned}
&\int Bes_n(x,y)\,\mathrm{d}y = Bes_{n-1}(x,y)\\
&\int \frac{(\beta\cdot y)^m}{m!}\,Bes_n(x,\delta y)\,\mathrm{d}y = \frac{1}{\delta}\left(\frac{\beta}{\delta}\right)^m\cdot\sum_{k=0}^{m}(-1)^{m-k}\cdot\frac{(\delta y)^k}{k!}\cdot Bes_{n-m-1+k}(x,\delta y), \quad \delta\neq 0\\
&\int \mathrm{e}^{\alpha y}\cdot Bes_n(x,y)\,\mathrm{d}y = -(-\alpha)^{n-1}\cdot\mathrm{e}^{\alpha y}\cdot Bs_n\left(-\frac{x}{\alpha},-\alpha y\right), \quad \alpha\neq 0\\
&\int y^n\,\mathrm{e}^{\alpha y}\,Bes_n(x,y)\,\mathrm{d}y = -(-\alpha)^{-(n+1)}\cdot x^n\cdot\mathrm{e}^{\alpha y}\cdot Bs_{-n}\left(-\frac{x}{\alpha},-\alpha y\right), \quad \alpha\neq 0,\quad n\geqslant 0 \quad\text{or}\quad xy\neq 0\\
&\int y\,\mathrm{e}^{\alpha y}\,Bes_0(x,y)\,\mathrm{d}y = \frac{1}{\alpha}\,\mathrm{e}^{\alpha y}\left[y\cdot Bes_0(x,y)-\frac{1}{\alpha}Bs_0\left(-\frac{x}{\alpha},-\alpha y\right)+\frac{x}{\alpha^2}\cdot Bs_{-1}\left(-\frac{x}{\alpha},-\alpha y\right)\right], \quad \alpha\neq 0
\end{aligned} \tag{12}$$

and

$$\begin{aligned}
&\int Bes_n(x,y)\,\mathrm{d}x = Bes_{n+1}(x,y)\\
&\int \frac{(\beta\cdot x)^m}{m!}\,Bes_n(\gamma x,y)\,\mathrm{d}x = \frac{1}{\gamma}\left(\frac{\beta}{\gamma}\right)^m\sum_{k=0}^{m}(-1)^{m-k}\cdot\frac{(\gamma x)^k}{k!}\,Bes_{n+m+1-k}(\gamma x,y), \quad \gamma\neq 0\\
&\int \mathrm{e}^{\alpha x}\,Bes_n(x,y)\,\mathrm{d}x = (-\alpha)^{-(n+1)}\cdot\mathrm{e}^{\alpha x}\cdot Bs_{n+1}\left(-\alpha x,-\frac{y}{\alpha}\right), \quad \alpha\neq 0\\
&\int x^n\,\mathrm{e}^{\alpha x}\,Bes_{-n}(x,y)\,\mathrm{d}x = y^n\cdot\int \mathrm{e}^{\alpha x}\cdot Bes_n(x,y)\,\mathrm{d}x, \quad \alpha\neq 0,\quad n\geqslant 0 \quad\text{or}\quad xy\neq 0
\end{aligned} \tag{13}$$

When subintegral functions are expressed by means of the $Bs_n(x, y)$ functions, the following formulae are valid

$$\begin{aligned}
&\int Bs_n(x,y)\,\mathrm{d}y = Bs_{n-1}(x,y)\\
&\int \frac{(\beta y)^m}{m!}\,Bs_n(x,\delta y)\,\mathrm{d}y = \frac{1}{\delta}\left(\frac{\beta}{\delta}\right)^m\sum_{k=0}^{m}(-1)^{m-k}\cdot\frac{(\delta y)^k}{k!}\,Bs_{n-m-1+k}(x,\delta y), \quad \delta\neq 0\\
&\int \mathrm{e}^{\alpha y}\,Bs_n(x,\delta y)\,\mathrm{d}y = \frac{1}{\delta}\mathrm{e}^{\alpha y}\sum_{j=0}^{\infty}\left(-\frac{\alpha}{\delta}\right)^j Bs_{n-1-j}(x,\delta y), \quad \delta\neq 0\\
&\int \mathrm{e}^{\alpha y}\,Bs_n(x,y)\,\mathrm{d}y = \begin{cases} -\mathrm{e}^{\alpha y}\cdot\displaystyle\sum_{k=n}^{\infty}(-\alpha)^{k-1}\cdot Bs_k\left(-\frac{x}{\alpha},-\alpha y\right), & \alpha\neq 0\\[2ex] \dfrac{\mathrm{e}^{\alpha y}}{1+\alpha}\left[Bs_n(x,y)-(-\alpha)^{n-1}Bs_n\left(-\dfrac{x}{\alpha},-\alpha y\right)\right], & \alpha\neq 0,\quad \alpha\neq -1\end{cases}
\end{aligned} \tag{14}$$

and

$$\begin{aligned}
&\int Bs_n(x,y)\,\mathrm{d}x = Bs_{n+1}(x,y)\\
&\int \frac{(\beta\cdot x)^m}{m!}\,Bs_n(\gamma x,y)\,\mathrm{d}x = \frac{1}{\gamma}\left(\frac{\beta}{\gamma}\right)^m\cdot\sum_{k=0}^{m}(-1)^{m-k}\cdot\frac{(\gamma x)^k}{k!}\cdot Bs_{n+m+1-k}(\gamma x,y), \quad \gamma\neq 0\\
&\int \mathrm{e}^{\alpha x}\,Bs_n(\gamma x,y)\,\mathrm{d}x = \frac{1}{\delta}\,\mathrm{e}^{\alpha x}\cdot\sum_{i=0}^{\infty}\left(-\frac{\alpha}{\gamma}\right)^i Bs_{n+1+i}(\gamma x,y), \quad \gamma\neq 0\\
&\int \mathrm{e}^{\alpha x}\,Bs_n(x,y)\,\mathrm{d}x = \begin{cases} \mathrm{e}^{\alpha x}\cdot\displaystyle\sum_{k=n}^{\infty}(-\alpha)^{-(k+1)}\cdot Bs_{k+1}\left(-\alpha x,-\frac{y}{\alpha}\right), & \alpha\neq 0\\[2ex] \dfrac{\mathrm{e}^{\alpha x}}{1+\alpha}\left[Bs_n(x,y)-(-\alpha)^{-(n+1)}Bs_n\left(-\alpha x,-\dfrac{y}{\alpha}\right)\right], & \alpha\neq 0,\quad \alpha\neq -1.\end{cases}
\end{aligned} \tag{15}$$

Making use of the differential formulae, equations (10) and (11), it is easy to derive the following two types of formulae

$$\begin{aligned} &\int Bes_n(x,\beta y)\cdot Bes_{n+1}(x,\beta y)\,\mathrm{d}y = \frac{1}{2\beta^{2n+1}}[Bes_n(\beta x, y)]^2, \quad \beta \neq 0 \\ &\int Bes_n(\alpha x, y)\cdot Bes_{n-1}(\alpha x, y)\,\mathrm{d}x = \frac{1}{2}\alpha^{2n-1}\cdot[Bes_n(x,\alpha y)]^2, \quad \alpha \neq 0 \end{aligned} \tag{16}$$

$$\begin{aligned} &\int Bs_n(x,\beta y)\cdot Bs_{n+1}(x,\beta y)\,\mathrm{d}y = \frac{1}{2\beta}[Bs_n(x,\beta y)]^2, \quad \beta \neq 0 \\ &\int Bs_n(\alpha x, y)\cdot Bs_{n-1}(\alpha x, y)\,\mathrm{d}x = \frac{1}{2\alpha}[Bs_n(\alpha x, y)]^2, \quad \alpha \neq 0. \end{aligned} \tag{17}$$

The properties presented above of the two families of functions make it possible to obtain the solutions of many technical problems in a 'closed' form although the properties above are not completed. In this paper, we confine ourselves to the analyses of the cross-flow recuperator and the heat transfer in the fixed clinker beds cooled by a gas medium.

Further properties of the $Bes_n(x, y)$ and $Bs_n(x, y)$ functions and their applications, in the theory of regenerators and heat exchangers in unsteady conditions, will be treated in another paper.

3. EXAMPLES OF APPLICATIONS OF THE $Bes_n(x, y)$ AND $Bs_n(x, y)$ FUNCTIONS IN THE HEAT EXCHANGERS THEORY

In this section we describe a number of practical applications of the heat transfer theory examined in Section 2 for the two families of functions $Bes_n(x, y)$ and $Bs_n(x, y)$.

3.1. *The cross-flow recuperator*

One of the simplest applications is the theory of the two-media cross-flow recuperator under Nusselt's assumptions [9, 10]. This problem was considered in ref. [6], where the exact solution, adduced below, was found.

It is known that the mathematical description of the cross-flow recuperator under the assumptions mentioned above reduces to the following system of two linear partial differential equations of the first order with constant coefficients

$$\begin{aligned} \frac{\partial T_1}{\partial \xi} &= -a\cdot T_1(\xi,\eta)+a\cdot T_2(\xi,\eta) \\ \frac{\partial T_2}{\partial \eta} &= b\cdot T_1(\xi,\eta)-b\cdot T_2(\xi,\eta) \end{aligned} \tag{18}$$

and the boundary conditions

$$\begin{cases} T_1(\xi = 0,\eta) = 1 \\ T_2(\xi,\eta = 0) = 0. \end{cases} \tag{19}$$

The final form of the solution was found [6] in the following form

$$\begin{cases} T_1(\xi,\eta) = 1-\exp[-(a\xi+b\eta)]\cdot Bs_1(a\xi,b\eta) \\ T_2(\xi,\eta) = 1-\exp[-(a\xi+b\eta)]\cdot Bs_0(a\xi,b\eta). \end{cases} \tag{20}$$

It is easy to verify that equations (20) fulfil equations (18) with boundary conditions (19). However, in the case when boundary conditions are defined by means of the exponential functions

$$\begin{cases} T_1(\xi = 0,\eta) = A_1\cdot e^{A_2\eta} \\ T_2(\xi,\eta = 0) = B_1\cdot e^{B_2\xi} \end{cases} \tag{21}$$

the solution of the balance equations (18) is given by

$$
\begin{aligned}
T_1(\xi,\eta) &= A_1\cdot\exp\left[-\left(1-\frac{b}{A_2+b}\right)a\cdot\xi+A_2\cdot\eta\right]+\exp\left[-(a\xi+b\eta)\right]\cdot\left\{\frac{aB_1}{B_2+a}\right.\\
&\quad\left.\times Bs_1\left[(B_2+a)\xi,\frac{ab}{B_2+a}\eta\right]-A_1\cdot Bs_1\left[\frac{ab}{A_2+b}\xi,(A_2+b)\eta\right]\right\}\\
T_2(\xi,\eta) &= \frac{bA_1}{A_2+b}\exp\left[-\left(1-\frac{b}{A_2+b}\right)a\xi+A_2\eta\right]+\exp\left[-(a\xi+b\eta)\right]\left\{B_1\cdot Bs_0\left[(B_2+a)\xi,\right.\right.\\
&\quad\left.\left.\times\frac{ab}{B_2+a}\eta\right]-\frac{bA_1}{A_2+b}\cdot Bs_0\left[\frac{ab}{A_2+b}\xi,(A_2+b)\eta\right]\right\}.
\end{aligned}
\tag{22}
$$

It is worth underlining that, if $A_1 = 1$ and $A_2 = B_1 = B_2 = 0$, then conditions (21) become equivalent to conditions (19) and solution (22) reduces to solution (20).

Let us consider such a case when the inlet fluid temperatures are linear functions of appropriate coordinates. It means, that

$$
\begin{cases}
T_1(\xi=0,\eta) = A_0+A_1\cdot\eta = W_1(\eta)\\
T_2(\xi,\eta=0) = B_0+B_1\xi = W_1(\xi).
\end{cases}
\tag{23}
$$

Based on the general solution [1, 5, 6] of system (18) one obtains

$$
\begin{aligned}
T_1(\xi,\eta) &= e^{-a\xi}\left\{W_1(\eta)\left[1+\int_0^\eta e^{-b\mu}\cdot Bes_1(ab\xi,\mu)\,d\mu\right]-A_1\cdot\int_0^\eta \mu\cdot e^{-b\mu}\cdot Bes_1(ab\xi,\mu)\,d\mu\right\}+a\cdot e^{-b\eta}\\
&\quad\times\left[W_1(\xi)\cdot\int_0^\xi e^{-a\mu}\cdot Bes_0(ab\eta,\mu)\,d\mu-B_1\cdot\int_0^\xi \mu\cdot e^{-a\mu}\cdot Bes_0(ab\eta,\mu)\,d\mu\right]\\
T_2(\xi,\eta) &= b\,e^{-a\xi}\left[W_1(\eta)\cdot\int_0^\eta e^{-b\mu}\cdot Bes_0(ab\xi,\mu)\,d\mu-A_1\int_0^\eta \mu e^{-b\mu}\cdot Bes_0(ab\xi,\mu)\,d\mu\right]+e^{-b\eta}\\
&\quad\times\left\{W_1(\xi)\left[1+\int_0^\xi e^{-a\mu}\cdot Bes_1(ab\eta,\mu)\,d\mu\right]-B_1\cdot\int_0^\xi \mu\cdot e^{-a\mu}\cdot Bes_1(ab\eta,\mu)\,d\mu\right\}.
\end{aligned}
\tag{24}
$$

We give this form of solution in order to pay attention to types of subintegral functions for which the primary functions should be found.

Making use of the formulae presented in the previous paragraph, one finally obtains

$$
\begin{aligned}
T_1(\xi,\eta) &= W_1(\eta)\cdot\left[1-e^{-(a\xi+b\eta)}\cdot Bs_1(a\xi,b\eta)\right]-\frac{a}{b}A_1\cdot\xi\left[1-e^{-(a\xi+b\eta)}\cdot Bs_{-1}(a\xi,b\eta)\right]+W_1(\xi)\cdot e^{-(a\xi+b\eta)}\\
&\quad\times Bs_1(a\xi,b\eta)+\frac{1}{a}B_1\cdot e^{-(a\xi+b\eta)}\cdot\left\{b\eta\left[Bes_0(a\xi,b\eta)-Bs_2(a\xi,b\eta)\right]-Bs_1(a\xi,b\eta)\right\}\\
T_2(\xi,\eta) &= W_1(\eta)\cdot\left[1-e^{-(a\xi+b\eta)}Bs_0(a\xi,b\eta)\right]-\frac{1}{b}A_1\left\{(1+a\xi)-e^{-(a\xi+b\eta)}\left[a\xi\cdot(Bes_0(a\xi,b\eta)\right.\right.\\
&\quad\left.\left.+Bs_{-1}(a\xi,b\eta))+Bs_0(a\xi,b\eta)\right]\right\}+W_1(\xi)\cdot e^{-(a\xi+b\eta)}\cdot Bs_0(a\xi,b\eta)-\frac{b}{a}\cdot B_1\cdot\eta\cdot e^{-(a\xi+b\eta)}\cdot Bs_2(a\xi,b\eta).
\end{aligned}
\tag{25}
$$

At present, we are going to generalize a form of the boundary conditions (23). We consider the case when these conditions are defined by means of polynomial functions of degree N and M, respectively

$$
\begin{aligned}
T_1(\xi=0,\eta) &= \sum_{i=0}^{N} A_i\cdot\eta^i = W_N(\eta)\\
T_2(\xi,\eta=0) &= \sum_{i=0}^{M} B_i\cdot\xi^i = W_M(\xi).
\end{aligned}
\tag{26}
$$

The solution then takes the form

$$\begin{aligned} T_1(\xi,\eta) &= e^{-a\xi}\left[W_N(\eta)+\sum_{i=0}^{N} A_i \sum_{k=0}^{i}(-1)^k\binom{i}{k}\eta^{i-k}\int_0^{\eta}\mu^k e^{-b\mu} Bes_1(ab\xi,\mu)\,d\mu\right] \\ &+a\cdot e^{-b\eta}\cdot\sum_{i=0}^{M} B_i\sum_{k=0}^{i}(-1)^k\binom{i}{k}\xi^{i-k}\int_0^{\xi}\mu^k e^{-a\mu}Bes_0(ab\eta,\mu)\,d\mu \\ T_2(\xi,\eta) &= e^{-b\eta}\left[W_M(\xi)+\sum_{i=0}^{M} B_i \sum_{k=0}^{i}(-1)^k\binom{i}{k}\xi^{i-k}\int_0^{\xi}\mu^k e^{-a\mu} Bes_1(ab\eta,\mu)\,d\mu\right] \\ &+b\cdot e^{-a\xi}\cdot\sum_{i=0}^{N} A_i\sum_{k=0}^{i}(-1)^k\binom{i}{k}\eta^{i-k}\int_0^{\eta}\mu^k e^{-b\mu}Bes_0(ab\xi,\mu)\,d\mu. \end{aligned} \tag{27}$$

Let us observe that, for $M = N = 0$ and $A_0 = 1, B_0 = 0$, the solution above becomes identical with equations (20) and, for $M = N = 1$, is identical with equations (24).

The following integrals occur in formulae (27)

$$\begin{aligned} \Psi_k(x,y) &= \int y^k e^{\alpha y} Bes_1(x,y)\,dy, \quad \alpha \neq 0 \\ \Phi_k(x,y) &= \int y^k e^{\alpha y} Bes_0(x,y)\,dy, \quad \alpha \neq 0. \end{aligned} \tag{28}$$

In the previous section, we have given formulae for calculation of the functions Ψ_k and Φ_k when $k = 0$ and 1. For any natural number $k \geqslant 1$, the following recurring formulae hold

$$\begin{aligned} \Psi_k(x,y) &= \frac{1}{\alpha}[y^k e^{\alpha y} Bes_1(x,y)-(k-1)\Psi_{k-1}(x,y)-x\Phi_{k-1}(x,y)], \quad \alpha \neq 0 \\ \Phi_k(x,y) &= \frac{1}{\alpha}[y^k e^{\alpha y} Bes_0(x,y)-k\cdot\Phi_{k-1}(x,y)-\Psi_k(x,y)], \quad \alpha \neq 0 \end{aligned} \tag{29}$$

where

$$\begin{aligned} \Psi_0(x,y) &= -e^{\alpha y}\cdot Bs_1\left(-\frac{x}{\alpha}, -\alpha y\right), \quad \alpha \neq 0 \\ \Phi_0(x,y) &= \frac{1}{\alpha}e^{\alpha y}\cdot Bs_0\left(-\frac{x}{\alpha}, -\alpha y\right), \quad \alpha \neq 0. \end{aligned} \tag{30}$$

Relations (29) and (30) are next used in formulae (27). We do not present possible transformations of solution (27) because formulae (27) and (30) constitute a sufficiently clear algorithm for eventual numerical calculations.

We call the reader's attention to the fact that, due to introducing the two families of $Bes_n(x, y)$ and $Bs_n(x, y)$ functions, finding solution (27) with boundary conditions (26) is available without refering to approximate analytical or numerical methods.

The presented examples do not exhaust all possible forms of boundary conditions which may be associated with system (18). However, one should note the fact that, the problem of finding a solution of this system of equations with every type of boundary condition may be reduced to calculation of appropriate integrals for which subintegral functions involve functions from the family (5). The knowledge of determining primary functions enables us to obtain the solutions in the so-called closed form. If boundary conditions are such a type that it is impossible analytically to calculate some integrals, occurring in the solution of system (18), or if determining of primary functions is not purposeful from practical reasons, then these integrals are calculated numerically.

Apart from this, such a method of determining a solution is better than approximate analytical and numerical methods because it leads to more precise results and less time is necessary for calculations. What is more, there does not exist the problem of convergence which is essential in approximate methods.

3.2. *The fixed clinker beds cooled by a gas medium*

Another example of applications of $Bes_n(x, y)$ and $Bs_n(x, y)$ functions is the problem of heat transfer in fixed clinker beds cooled by a gas medium. Necessity of its solution appears during the determination of the volumetric heat transfer coefficient, which is one of the basic parameters defining physical properties of the beds. Theoretical and experimental analysis reduces to the exploration of unsteady thermal conditions caused by an appropriate heat source located in the control plane in front of the beds.

Under some simplifying assumptions, this problem consists of finding a solution of the differential equations system describing the energy conservation of the beds and a gas medium [1]

$$\begin{aligned} \frac{\partial \theta_1}{\partial \tau} &= \theta_2 - \theta_1 \\ \mu \frac{\partial \theta_2}{\partial \tau} + \frac{\partial \theta_2}{\partial \xi} &= \theta_1 - \theta_2 \end{aligned} \tag{31}$$

with the initial boundary conditions

$$\begin{cases} \theta_1(\xi, \mu\xi) = 0 \\ \theta_2(0, \tau) = f(\tau). \end{cases} \tag{32}$$

This problem was solved in ref. [1] under the assumption

$$f(\tau) = 1 - e^{-\kappa\tau}, \quad \kappa \neq 1. \tag{33}$$

In this case, the solution is given by

$$\begin{aligned} \theta_1(\xi, \tau) &= 1 + \exp\{-[(1-\mu)\xi + \tau]\}\left\{\frac{1}{1-\kappa} Bs_0\left[\frac{1}{1-\kappa}\xi, (1-\kappa)(\tau - \mu\xi)\right]\right. \\ &\qquad \left. - Bs_0(\xi, \tau - \mu\xi)\right\} - \frac{1}{1-\kappa}\exp\left\{\kappa\left[\frac{1+\mu(1-\kappa)}{1-\kappa}\xi - \tau\right]\right\} \\ \theta_2(\xi, \tau) &= 1 + \exp\{-[(1-\mu)\xi + \tau]\}\left\{Bs_1\left[\frac{1}{1-\kappa}\xi, (1-\kappa)(\tau - \mu\xi)\right]\right. \\ &\qquad \left. - Bs_1(\xi, \tau - \mu\xi)\right\} - \exp\left\{\kappa\left[\frac{1+\mu(1-\kappa)}{1-\kappa}\xi - \tau\right]\right\}. \end{aligned} \tag{34}$$

The solution for the case when $\kappa = 1$, omitted in ref. [1], is given in the Appendix. Now, we consider the case when the initial condition is as follows

$$f(\tau) = 1 - A_1 \cdot \tau. \tag{35}$$

Below, the method of finding a solution for the differential equations systems (3) described in ref. [1], is applied. Observe that system (31) is a particular case of system (3). Since in the case considered the matrix $[c_{ki}]$ is not singular, the system above may be reduced to the form (2).

The transformation of variables

$$\begin{cases} \xi^* = \xi \\ \tau^* = \tau - \mu\xi \end{cases} \tag{36}$$

yields

$$\begin{aligned} \frac{\partial \theta_1^*}{\partial \tau^*} &= \theta_2^* - \theta_1^* \\ \frac{\partial \theta_2^*}{\partial \xi^*} &= \theta_1^* - \theta_2^* \end{aligned} \tag{37}$$

where

$$\theta_i(\xi, \tau) \overset{\text{df}}{=} \theta_i^*(\xi^*, \tau^*) = \theta_i^*(\xi, \tau - \mu\xi). \tag{38}$$

The following initial boundary conditions

$$\begin{cases} \theta_1^*(\xi^*, 0) = 0 \\ \theta_2^*(0, \tau^*) = f(\tau^*) \end{cases} \tag{39}$$

are associated with system (37).

The problem of unsteady heat transfer in the beds was reduced to finding of a solution of the same differential equations system as in the case of the cross-flow recuperator. For these reasons, we pay attention to those elements which were omitted above when determining solutions of system (18) and to the inverse transformation of variables.

The solution of system (37) with the initial boundary conditions (39) may be expressed as

$$\begin{aligned} \theta_1^*(\xi^*,\tau^*) &= e^{-\xi^*}\cdot\int_0^{\tau^*} f(\tau^*-\eta^*)\cdot e^{-\eta^*}\cdot Bes_0(\xi^*,\eta^*)\,d\eta^* \\ \theta_2^*(\xi^*,\tau^*) &= e^{-\xi^*}\cdot\left[f(\tau^*)+\int_0^{\tau^*} f(\tau^*-\eta^*)\cdot e^{-\eta^*}\cdot Bes_1(\xi^*,\eta^*)\,d\eta^*\right]. \end{aligned} \tag{40}$$

After the inverse transformation of variables, we arrive at

$$\begin{aligned} \theta_1(\xi,\tau) &= e^{-\xi}\cdot\int_0^{\tau-\mu\xi} f(\tau-\mu\xi-\eta)e^{-\eta}\cdot Bes_0(\xi,\eta)\,d\eta \\ \theta_2(\xi,\tau) &= e^{-\xi}\left[f(\tau-\mu\xi)+\int_0^{\tau-\mu\xi} f(\tau-\mu\xi-\eta)\cdot e^{-\eta}\cdot Bes_1(\xi,\eta)\,d\eta\right]. \end{aligned} \tag{41}$$

It is interesting to note that the important relation

$$\left.\frac{\partial\theta_2^*}{\partial\tau^*}\right|_{\tau^*=0} = e^{-\xi^*}\cdot\left.\frac{df}{d\tau^*}\right|_{\tau^*=0} \tag{42}$$

instantly results from equations (40). This fact was observed in ref. [1]. This formula may be used when elaborating a method of measurement of the volumetric heat transfer coefficient.

Since

$$\begin{aligned} \Psi_1(\xi^*,\tau^*) &= -\xi^*\cdot e^{-\tau^*}\cdot Bs_{-1}(\xi^*,\tau^*) \\ \Phi_1(\xi^*,\tau^*) &= -e^{-\tau^*}[\tau^*\cdot Bes_0(\xi^*,\tau^*)+Bs_0(\xi^*,\tau^*)+\xi^* Bs_1(\xi^*,\tau^*)] \end{aligned} \tag{43}$$

and

$$\begin{cases} \Psi_1(\xi^*,0) = -\xi^*\cdot e^{\xi^*} \\ \Phi_1(\xi^*,0) = -(1+\xi^*)e^{\xi^*} \end{cases} \tag{44}$$

solution (40) with initial boundary conditions (35) may be written as

$$\begin{aligned} \theta_1^*(\xi^*,\tau^*) &= (1-A_1\cdot\tau^*)\cdot[1-e^{-(\xi^*+\tau^*)}Bs_0(\xi^*,\tau^*)]-A_1\{e^{-(\xi^*+\tau^*)} \\ &\quad\times[\tau^*\cdot Bes_0(\xi^*,\tau^*)+Bs_0(\xi^*,\tau^*)+\xi^* Bs_{-1}(\xi^*,\tau^*)]-(1+\xi^*)\} \\ \theta_2^*(\xi^*,\tau^*) &= (1-A_1\cdot\tau^*)[1-e^{-(\xi^*+\tau^*)}Bs_1(\xi^*,\tau^*)]+A_1\cdot\xi^*[1-e^{-(\xi^*+\tau^*)}Bs_{-1}(\xi^*,\tau^*)]. \end{aligned} \tag{45}$$

Next, after the inverse transformation, these expressions may be rewritten in the form

$$\begin{aligned} \theta_1(\xi,\tau) &= [1-A_1\cdot(\tau-\mu\xi)]\{1-e^{-[(1-\mu)\xi+\tau]}Bs_0(\xi,\tau-\mu\xi)\}-A_1\{e^{-[(1-\mu)\xi+\tau]}\cdot[(\tau-\mu\xi) \\ &\quad\times Bes_0(\xi,\tau-\mu\xi)+Bs_0(\xi,\tau-\mu\xi)+\xi\cdot Bs_{-1}(\xi,\tau-\mu\xi)]-(1+\xi)\} \\ \theta_2(\xi,\tau) &= [1-A_1\cdot(\tau-\mu\xi)]\{1-e^{-[(1-\mu)\xi+\tau]}Bs_1(\xi,\tau-\mu\xi)\}+A_1\xi\{1-e^{-[(1-\mu)\xi+\tau]}Bs_{-1}(\xi,\tau-\mu\xi)\}. \end{aligned} \tag{46}$$

If unsteady thermal conditions in the system: fixed clinker beds–gas medium are caused by such temperature changes in the inlet clinker beds section which may be approximated by the function

$$f(\tau) = 1-\sum_{i=1}^{N} A_i\cdot\tau^i \tag{47}$$

then solution (40) takes the form

$$\begin{aligned} \theta_1^*(\xi^*,\tau^*) &= [1-e^{-(\xi^*+\tau^*)}Bs_0(\xi^*,\tau^*)]\cdot\left(1-\sum_{i=1}^{N} A_i\cdot\tau^{*i}\right)-A_1\{e^{-(\xi^*+\tau^*)} \\ &\quad\times[\tau^*\cdot Bes_0(\xi^*,\tau^*)+Bs_0(\xi^*,\tau^*)-\xi^* Bs_{-1}(\xi^*,\tau^*)]-(1+\xi^*)\}+Q_1(\xi^*,\tau^*) \\ \theta_2^*(\xi^*,\tau^*) &= [1-e^{-(\xi^*+\tau^*)}Bs_1(\xi^*,\tau^*)]\cdot\left(1-\sum_{i=1}^{N} A_i\cdot\tau^{*i}\right)+A_1\cdot\xi^*[1-e^{-(\xi^*+\tau^*)}Bs_{-1}(\xi^*,\tau^*)]+Q_2(\xi^*,\tau^*) \end{aligned} \tag{48}$$

where

$$\begin{aligned} Q_1(\xi^*,\tau^*) &= -e^{-\xi^*}\sum_{i=2}^{N} A_i\sum_{k=1}^{i}(-1)^k\binom{i}{k}\tau^{*i-k}[\Phi_k(\xi^*,\tau^*)-\Phi_k(\xi^*,0)] \\ Q_2(\xi^*,\tau^*) &= -e^{-\xi^*}\sum_{i=2}^{N} A_i\sum_{k=1}^{i}(-1)^k\binom{i}{k}\tau^{*i-k}[\Psi_k(\xi^*,\tau^*)-\Psi_k(\xi^*,0)]. \end{aligned} \tag{49}$$

Taking into account expressions (29) and the formula

$$\bigwedge_i \sum_{k=1}^{i} (-1)^k \binom{i}{k} = -1 \tag{50}$$

one obtains

$$\begin{aligned} Q_1(\xi^*,\tau^*) &= -e^{-(\xi^*+\tau^*)}[Bes_0(\xi^*,\tau^*)+Bes_1(\xi^*,\tau^*)]\sum_{i=2}^{N} A_i\cdot\tau^{*i} - e^{-\xi^*}\sum_{i=2}^{N} A_i \sum_{k=1}^{i}(-1)^k\binom{i}{k}\tau^{*i-k}\{(k-1)\\ &\quad\times[\psi_{k-1}(\xi^*,\tau^*)-\psi_{k-1}(\xi^*,0)]+(k+\xi^*)[\Phi_{k-1}(\xi^*,\tau^*)-\Phi_{k-1}(\xi^*,0)]\}\\ Q_2(\xi^*,\tau^*) &= -e^{-(\xi^*+\tau^*)}\cdot Bes_0(\xi^*,\tau^*)\sum_{i=2}^{N} A_i\cdot\tau^{*i} - e^{-\xi^*}\cdot\sum_{i=2}^{N} A_i\sum_{k=1}^{i}(-1)^k\binom{i}{k}\\ &\quad\times\tau^{*i-k}\cdot\{\xi^*\cdot[\Phi_{k-1}(\xi^*,\tau^*)-\Phi_{k-1}(\xi^*,0)]+(k-1)\cdot[\Psi_{k-1}(\xi^*,\tau^*)-\Psi_{k-1}(\xi^*,0)]\}. \end{aligned} \tag{51}$$

Next, these relations should be taken into consideration in solution (48). The inverse transformation is carried out in the same way as in the two previous examples.

Observe that, if $A_2 = A_3 = \cdots = A_N = 0$, then the latest solution becomes identical with expressions (45). If, in addition, $A_1 = 0$ then this solution becomes identical with equations (20). An analogous analysis may also be carried out in the case of solution (27).

4. FINAL REMARKS

(1) Special attention should be paid to one of the identities (7). This identity determines the relation between Bes_n functions and nth order first kind ordinary and modified Bessel functions J_n and I_n. Making use of intermediate results [6] obtained in the course of solving the problem (18) and (19), one can write solution (20) in the form

$$\begin{aligned} T_1(\xi,\eta) &= e^{-a\xi}\cdot\left[1+\int_0^\eta e^{-b\mu}\cdot Bes_1(ab\xi,\mu)\,d\mu\right]\\ T_2(\xi,\eta) &= b\cdot e^{-a\xi}\cdot\int_0^\eta e^{-b\mu}\cdot Bes_0(ab\xi,\mu)\,d\mu. \end{aligned} \tag{52}$$

For considered values of arguments, the Bes_n functions are related to modified Bessel functions I_n. Taking into consideration the foregoing mentioned identity in solution (52), one obtains

$$\begin{aligned} T_1(\xi,\eta) &= e^{-a\xi}\cdot\left\{1+(ab\xi)^{1/2}\cdot\int_0^\eta e^{-b\mu}\cdot I_1[2(ab\xi\mu)^{1/2}]\mu^{-1/2}\,d\mu\right\}\\ T_2(\xi,\eta) &= b\cdot e^{-a\xi}\cdot\int_0^\eta e^{-b\mu}\cdot I_0[2(ab\xi\mu)^{1/2}]\,d\mu. \end{aligned} \tag{53}$$

In such forms [4, 11] are cited solutions of various technical problems described by the system of differential equations of the form (2) or (3). However, let us note that making use of the definition

$$Bs_n(x,y) = \sum_{k=n}^{\infty} Bes_k(x,y) \tag{54}$$

and using as well the previously discussed identity, one obtains solution (20) as well as solution (53) in the following form

$$\begin{aligned} T_1(\xi,\eta) &= 1-e^{-(a\xi+b\eta)}\sum_{k=1}^{\infty}\left(\frac{a\xi}{b\eta}\right)^{k/2} I_k[2(ab\xi\eta)^{1/2}]\\ T_2(\xi,\eta) &= 1-e^{-(a\xi+b\eta)}\sum_{k=0}^{\infty}\left(\frac{a\xi}{b\eta}\right)^{k/2} I_k[2(ab\xi\eta)^{1/2}]. \end{aligned} \tag{55}$$

Similar transformations can be also done for the remaining solutions written in the preceding section. In refs. [1, 2, 5, 6], the relation between the family of Bes_n functions and the Bessel functions was not exposed properly.

(2) It seems that solving differential equations (2) or (3) using $Bes_n(x, y)$ and $Bs_n(x, y)$ functions is much more effective than that based on the so-called fundamental function [4, 12, 13] of the form

$$J(x,y) = 1-e^{-y}\cdot\int_0^x e^{-\mu}\cdot I_0[2(\mu y)^{1/2}]\,d\mu \tag{56}$$

or

$$\phi(x, y) = [1 - J(x, y)]\, e^{x+y}. \tag{57}$$

(3) In the scope of the problems contained here ref. [14] is worth a note. Montakhab obtained in principle the identical solution of the problem which is equivalent to equations (18) and (19) or equations (31) and (32) in a different way. However, he restricted his attention to the analysis of one particular case, not trying the generalization presented in refs. [1, 2, 5, 6] and also in the present work.

(4) It should be stated that it is possible to obtain the solution in the closed form of differential equation systems in the form of equation (2) or equation (3) with various boundary or initial boundary conditions. Therefore, there is no need to use approximate methods. Only in cases of especially complicated forms of boundary or initial boundary conditions is it necessary to calculate some integrals numerically. However, it is a much easier task than numerically solving systems of equations of the form of equation (2) or equation (3).

(5) The systems of differential equations considered in this paper have the differential form of energy balances and are rather commonly used in the theory of heat transfer and in chemical engineering. The assumptions made in the course of simplifying the full energy balances can be accepted in the analysis of many physical phenomena and chemical processes. This fact proves the potential possibilities of the proposed method of obtaining the solutions and of the $Bes_n(x, y)$ and $Bs_n(x, y)$ functions.

(6) The relation between Bes_n functions and Bessel functions J_n and I_n enables one to obtain many primary functions for subintegral expressions containing the mentioned Bessel functions. The majority of results obtained in this way is not given in the commonly known and highly estimated work [15].

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REFERENCES

1. J. Łach et W. Pieczka, Une certaine méthode de calcul des échangeurs de chaleur, *Int. J. Heat Mass Transfer* **26**, 891–899 (1984).
2. J. Łach et W. Pieczka, Régénérateur avec la répartition sinusoïdale de la temperature d'éntrée du fluide, *Int. J. Heat Mass Transfer*, in press (1984).
3. D. P. Campbell, *Process Dynamics*. Wiley, New York (1958); Polish transl., PWN, Warszawa, pp. 215–278 (1962).
4. H. S. Carslaw and J. C. Jaeger, *Conduction of Heat in Solids* (Russian transl.), pp. 385–392. Izdat. Nauka, Moscow (1964).
5. W. Pieczka, On the general solution of some partial differential equations in two-variables domain, *Demonstratio Mathematica* **VIII**(4), 451–462 (1974).
6. J. Łach, The exact solution of the Nusselt's model of the cross-flow recuperator, *Int. J. Heat Mass Transfer* **26**, 1597–1601 (1984).
7. J. Mikusiński, *Rachunek Operatorów* (*Operational Calculus*). Polskie Towarzystwo Matematyczne (Polish Mathematical Society), Warszawa (1953).
8. L. S. Bark and P. I. Kuznetsov, *Tables of the Cylindrical Functions of the Two Imaginary Variables* (in Russian). Moscow (1962).
9. W. Nusselt, Der Wärmeübergang im Kreuzstrom, *Z. Ver. Dt. Ing.* 2021 (1911).
10. W. Nusselt, Eine neue Formel für den Wärmedurchgang im Kreuzstrom, *Tech. Mech. Thermo-Dynam., Berl.* **1**(12), 417–422 (1930).
11. D. L. Briggs and C. N. Shen, Distributed parameter optimum control of a nuclear rocket with thermal stress constraints, *Trans. Am. Soc. Mech. Engrs*, Series D, *J. Basic Engng*, 300–306 (1967).
12. S. Goldstein, On the mathematics of exchange processes in fixed columns, *Proc. R. Soc.* **219A**, 151–185 (1953).
13. H. C. Thomas, Heterogeneous ion exchange in a flowing system, *J. Am. Chem. Soc.* **66**, 1664–1666 (1944).
14. A. Montakhab, Convective heat transfer in porous media, *Trans. Am. Soc. Mech. Engrs*, Series C, *J. Heat Transfer* **101**, 507–510 (1979).
15. I. S. Gradstein and I. M. Rhizik, *Tables of Integrals, Sums, Series and Products* (Russian). Izdat. Nauka, Moscow (1971).

APPENDIX

If we set $\kappa = 1$ in the initial condition (33), then the solution given by formulae (34) possesses singularity which may be eliminated by taking the limits

$$\theta_i(\xi, \tau)|_{\kappa=1} = \lim_{\kappa \to 1} \theta_i(\xi, \tau), \quad i = 1, 2. \tag{A1}$$

In order to show this, solution (34) is rewritten in the form

$$\begin{aligned} \theta_1(\xi, \tau) &= 1 - \exp\{-[(1-\mu)\xi + \tau]\} \cdot Bs_0(\xi, \tau - \mu\xi) + P_1(\xi, \tau) \\ \theta_2(\xi, \tau) &= 1 - \exp\{-[(1-\mu)\xi + \tau]\} \cdot Bs_1(\xi, \tau - \mu\xi) + P_2(\xi, \tau) \end{aligned} \tag{A2}$$

where

$$
\begin{aligned}
P_1(\xi,\tau) &= \frac{1}{1-\kappa}\exp\{-[(1-\mu)\xi+\tau]\}\cdot Bs_0\left[\frac{1}{1-\kappa}\xi,(1-\kappa)(\tau-\mu\xi)\right]-\frac{1}{1-\kappa}\exp\left\{\kappa\left[\frac{1+\mu(1-\kappa)}{1-\kappa}\xi-\tau\right]\right\}\\
P_2(\xi,\tau) &= \exp\{-[(1-\mu)\xi+\tau]\}\cdot Bs_1\left[\frac{1}{1-\kappa}\xi,(1-\kappa)(\tau-\mu\xi)\right]-\exp\left\{\kappa\left[\frac{1+\mu(1-\kappa)}{1-\kappa}\xi-\tau\right]\right\}.
\end{aligned}
\tag{A3}
$$

In the sequel we shall only transform terms P_i, $i = 1, 2$. First, we write them in the form

$$
\begin{aligned}
P_1(\xi,\tau) &= -\exp\{-[(1-\mu)\xi+\tau]\}\cdot\frac{1}{1-\kappa}\cdot\left\{\exp\left[\frac{1}{1-\kappa}\xi+(1-\kappa)(\tau-\mu\xi)\right]-Bs_0\left[\frac{1}{1-\kappa}\xi,(1-\kappa)(\tau-\mu\xi)\right]\right\}\\
P_2(\xi,\tau) &= -\exp\{-[(1-\mu)\xi+\tau]\}\cdot\left\{\exp\left[\frac{1}{1-\kappa}\xi+(1-\kappa)(\tau-\mu\xi)\right]-Bs_1\left[\frac{1}{1-\kappa}\xi,(1-\kappa)(\tau-\mu\xi)\right]\right\}.
\end{aligned}
\tag{A4}
$$

Applying in turn:

(1) one of identities (8), namely

$$e^{x+y} \equiv Bs_n(x,y)+Bs_{-(n-1)}(y,x),$$

(2) definition (5) in the form

$$Bs_n(x,y)-Bs_{n+1}(x,y) = Bes_n(x,y),$$

(3) one of identities (7), namely

$$\alpha^n\cdot Bes_n(x,y) \equiv Bes_n\left(\alpha x,\frac{y}{\alpha}\right),\quad \alpha\neq 0$$

one obtains

$$
\begin{aligned}
P_1(\xi,\tau) &= -\exp\{-[(1-\mu)\xi+\tau]\}\cdot\left\{Bes_{-1}(\xi,\tau-\mu\xi)-\frac{1}{1-\kappa}Bs_2\left[(1-\kappa)(\tau-\mu\xi),\frac{1}{1-\kappa}\xi\right]\right\}\\
P_2(\xi,\tau) &= -\exp\{-[(1-\mu)\xi+\tau]\}\cdot\left\{Bes_0(\xi,\tau-\mu\xi)-Bs_1\left[(1-\kappa)(\tau-\mu\xi),\frac{1}{1-\kappa}\xi\right]\right\}.
\end{aligned}
\tag{A5}
$$

We have the following relations

$$
\begin{aligned}
\lim_{\kappa\to 1}\left\{\frac{1}{1-\kappa}\cdot Bs_2\left[(1-\kappa)(\tau-\mu\xi),\frac{1}{1-\kappa}\xi\right]\right\} &= \lim_{\kappa\to 1}\sum_{m=2}^{\infty}\frac{(\tau-\mu\xi)^m}{m!}(1-\kappa)^{m-1}\sum_{k=0}^{m-2}\frac{\xi^k}{k!}\cdot\frac{1}{(1-\kappa)^k} = 0\\
\lim_{\kappa\to 1} Bs_1\left[(1-\kappa)(\tau-\mu\xi),\frac{1}{1-\kappa}\xi\right] &= \lim_{\kappa\to 1}\sum_{m=1}^{\infty}\frac{(\tau-\mu\xi)^m}{m!}(1-\kappa)^m\sum_{k=0}^{m-1}\frac{\xi^k}{k!}\cdot\frac{1}{(1-\kappa)^k} = 0
\end{aligned}
\tag{A6}
$$

because the value of k is always less than m, $\bigwedge_m \max(k) < m$. Consequently, the final forms of terms P_1, P_2 are

$$
\begin{aligned}
P_1(\xi,\tau) &= -\exp\{-[(1-\mu)\xi+\tau]\}\cdot Bes_{-1}(\xi,\tau-\mu\xi)\\
P_2(\xi,\tau) &= -\exp\{-[(1-\mu)\xi+\tau]\}\cdot Bes_0(\xi,\tau-\mu\xi).
\end{aligned}
\tag{A7}
$$

Substituting relations (7) into expressions (1), we obtain the solution for the $\kappa = 1$ case in the form

$$
\begin{aligned}
\theta_1(\xi,\tau) &= 1-\exp\{-[(1-\mu)\xi+\tau]\}\cdot[Bs_0(\xi,\tau-\mu\xi)+Bes_{-1}(\xi,\tau-\mu\xi)]\\
\theta_2(\xi,\tau) &= 1-\exp\{-[(1-\mu)\xi+\tau]\}\cdot[Bs_1(\xi,\tau-\mu\xi)+Bes_0(\xi,\tau-\mu\xi)].
\end{aligned}
\tag{A8}
$$

The same result may be obtained by straightforward checking from formulae (41).

PROPRIÉTÉS DES CERTAINES FONCTIONS SPÉCIALES ALLIÉES AVEC LES FONCTIONS DE BESSEL ET LEURS APPLICATION EN THÉORIE DES ÉCHANGEURS DE CHALEUR

Résumé—On a présenté les propriétés de base, alliées avec les fonctions de Bessel, des deux familles de fonctions $Bes_n(x, y)$ et $Bs_n(x, y)$ lesquelles sont définies à l'aide des séries entières:

$$Bs_n(x,y) \overset{\text{df}}{=} \sum_{m=\max(0,n)}^{\infty} \frac{x^m}{m!} \sum_{k=0}^{m-n} \frac{y^k}{k!}$$

$$Bes_n(x,y) \overset{\text{df}}{=} \sum_{m=\max(-n,0)}^{\infty} \frac{x^{m+n}y^m}{(m+n)!m}.$$

On a donné les nombreuses formules qui sont de la forme d'identités, les règles de dérivation et les formules pour calculer des intégrales desquelles les fonctions sous-intégrales comprennent les séries entières citées ci-dessus. Puis, on a démontré quelques applications des fonctions $Bs_n(x, y)$ et $Bes_n(x, y)$ en théorie d'un récupérateur à courants croisés avec brassage et d'un gisement immobile de clinkier qui est refroidi par un gaz. On a considéré quelques cas des conditions aux limites et conditions spatio-temporelles. Les résolutions obtenues sont exprimées sous la forme analytique. Cette forme a une supériorité évidente sur les solutions approximatives. Les fonctions $Bs_n(x, y)$ et $Bes_n(x, y)$ peuvent être appliquées en analyse théorique de différents types des échangeurs de chaleur et de masse, d'appareils utilisés en génie chimique et en théorie de réglage et protection de considérable nombre des processus physiques.

DIE ANWENDUNG DER SPEZIELLEN MIT BESSELFUNKTIONEN VERWANDTEN FUNKTIONEN IN DER WÄRMEAUSTAUSCHERTHEORIE

Zusammenfassung—In der vorliegenden Arbeit sind grundlegende Eigenschaften der speziellen Funktionen $Bs_n(x, y)$ und $Bes_n(x, y)$, die mit Bessel-Funktionen verwandt sind, dargestellt. Die erwähnten Funktionen sind mittels der Potenzreihen von zwei Veränderlichen definiert. Die entsprechenden Potenzreihen sind von der Form:

$$Bs_n(x,y) \overset{\text{df}}{=} \sum_{m=\max(0,n)}^{\infty} \frac{x^m}{m!} \sum_{k=0}^{m-n} \frac{y^k}{k!}$$

$$Bes_n(x,y) \overset{\text{df}}{=} \sum_{m=\max(-n,0)}^{\infty} \frac{x^{m+n}y^m}{(m+n)!m!}.$$

Es sind dabei zahlreiche Formeln für die Identitäten, die Formeln für die Differentiation als auch die Formeln für die Integration der erwähnten Potenzreihen angegeben. Um die Anwendung dieser vorgeschlagenen Potenzreihen zu erläutern, sind einige Beispiele, wie z.B. der Kreuzstrom-Wärmeaustauscher und der Klinkerkühler, durchgerechnet. In der Betrachtung sind verschiedene Rand- und Anfangs-Bedingungen angenommen. Die Lösung dieser Aufgaben ist in geschlossener Form dargestellt, was besonders im Vergleich zu den angenäherten analytischen und numerischen Lösungen vorteilhaft ist. Die Funktionen $Bs_n(x, y)$ und $Bes_n(x, y)$ können auch in der theoretischen Analyse zahlreicher Wärmeaustauscher, Regeneratoren und chemischer Apparate angewandt werden. Unabhängig davon können die oben definierten Funktionen in der Regeltechnik zahlreicher technischen Aufgaben, die in der Physikochemie auftreten, behilflich sein.

О СВОЙСТВАХ НЕКОТОРЫХ СПЕЦИАЛЬНЫХ ФУНКЦИЙ, СВЯЗАННЫХ С ФУНКЦИЯМИ БЕССЕЛЯ, И ИХ ПРИМЕНЕНИИ ПРИ РАСЧЕТЕ ТЕПЛООБМЕННИКОВ

Аннотация—Рассмотрены основные свойства функций $Bs_n(x, y)$ и $Bes_n(x, y)$, связанных с функциями Бесселя и определяемых двойными степенными рядами:

$$Bs_n(x, y) \overset{\text{df}}{=} \sum_{m=\max(0,n)}^{x} \frac{x^m}{m!} \sum_{k=0}^{m-n} \frac{y^k}{k!}$$

$$Bes_n(x, y) \overset{\text{df}}{=} \sum_{m=\max(-n,0)}^{x} \frac{x^{m+n}y^m}{(m+n)!\,m!}.$$

Приведены многочисленные тождества, дифференциальные равенства и формулы для расчета интегралов, подынтегральные выражения которых включают указанные выше специальные функции. Также показано применение функций $Bs_n(x, y)$ и $Bes_n(x, y)$ при расчете рекуперативных теплообменников с перекрестным потоком и анализе теплопереноса охлаждаемых газом слоев шлака. Рассмотрено несколько случаев граничных и начальных условий. Решения, представленные в так называемом замкнутом виде, лучше, чем полученные приближенными методами. Функции $Bs_n(x, y)$ и $Bes_n(x, y)$ можно использовать для анализа тепло-и массообменников, регенераторов, ионообменников и различного рода тяжелого оборудования, применяемого в химической промышленности. Они могут также использоваться для контроля различных физических процессов.